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Existence of the Limit Value of Two Person Zero-Sum Discounted Repeated Games via Comparison Theorems

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Abstract We give new proofs of existence of the limit of the discounted values for two person zero-sum games in the three following frameworks: absorbing, recursive, incomplete information. The idea of these new proofs is to use some comparison criteria.

Keywords Stochastic games · Repeated games · Incomplete information · Asymptotic value · Comparison principle · Variational inequalities

1 Introduction

The purpose of this article is to present a unified approach to the existence of the limit value for two person zero-sum discounted games. The main tools used in the proofs are

- the fact that the discounted value satisfies the Shapley equation [1],
- properties of accumulation points of the discounted values, and of the corresponding optimal strategies,

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– comparison of two accumulation points leading to uniqueness and characterization.

We apply this program for three well known classes of games, each time covering the case where action spaces are compact.

For absorbing games, the results are initially due to Kohlberg [2] for finitely many actions, later extended in Rosenberg and Sorin [3] for the compact case. An explicit formula for the limit was recently obtained in Laraki [4], and we obtain a related one. The case of recursive games was first handled in Everett [5], with a different notion of limit value involving asymptotic payoff on plays. It was later shown by Sorin [6] that these results implied also the existence of the limit value for two person zero-sum discounted games. The last class corresponds to games with incomplete information, where the results were initially obtained in Aumann and Maschler [7] and Mertens and Zamir [8] (including also the asymptotic study of the finitely repeated games). In that case, we follow a quite similar approach to Laraki [9].

2 Model, Notations and Basic Lemmas

Let G be a two person zero-sum stochastic game defined by a finite state space Ω, compact metric action spaces I and J for player 1 and 2 (with mixed extensions $X = \Delta(I)$ and $Y = \Delta(J)$, respectively, where for a compact metric space C, $\Delta(C)$ denotes the set of Borel probabilities on C, endowed with the weak-$\star$ topology), a separately continuous real bounded payoff g on $I \times J \times \Omega$ and a separately continuous transition ρ from $I \times J \times \Omega$ to $\Delta(\Omega)$.

The game is played in discrete time. At stage t, given the state ω_t, the payers choose moves $i_t \in I$, $j_t \in J$, the stage payoff is $g_t = g(i_t, j_t, \omega_t)$ and the new state ω_{t+1} is selected according to $\rho(i_t, j_t, \omega_t)$, and is announced to the players. Given $\lambda \in]0, 1]$, the total evaluation in the λ-discounted game is $\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} g_t$.

The Shapley operator $\Phi(\lambda, f)$ [1] is then defined, for $\lambda \in [0, 1]$ and f in some closed subset $\mathcal{F}_0$ of the set of bounded functions from Ω to $\mathbb{R}$, by the formula

$$\begin{aligned}\Phi(\lambda, f)(\omega) &= \min_Y \max_X \bigl\{\lambda g(x, y, \omega) + (1-\lambda)\mathsf{E}_{\rho(x,y,\omega)} f(\cdot)\bigr\}\\ &= \max_X \min_Y \bigl\{\lambda g(x, y, \omega) + (1-\lambda)\mathsf{E}_{\rho(x,y,\omega)} f(\cdot)\bigr\},\end{aligned}$$

where g and ρ are bilinearly extended to $X \times Y$. For $\lambda > 0$, the only fixed point of $\Phi(\lambda, \cdot)$ is the value v_λ of the discounted game [1].

The sets of optimal actions of each player in the above formula are denoted by $X_\lambda(f)(\omega)$ and $Y_\lambda(f)(\omega)$. Let $\mathbf{X} = X^\Omega$ and, similarly, $\mathbf{Y} = Y^\Omega$. For simplicity, for any $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$ we denote $\rho(\mathbf{x}, \mathbf{y}, \omega) := \rho(\mathbf{x}(\omega), \mathbf{y}(\omega), \omega)$. Moreover, define $\mathbf{X}_\lambda(f) := \prod_{\omega \in \Omega} X_\lambda(f)(\omega)$ and $\mathbf{Y}_\lambda(f) := \prod_{\omega \in \Omega} Y_\lambda(f)(\omega)$.

$\mathcal{S}$ denotes the set of fixed points of the projective operator $\Phi(0, .)$, and $\mathcal{S}_0$ is the set of accumulation points of the family $\{v_\lambda\}$ as λ goes to 0.

The following lemmas are easy to establish in this finite state framework.

Lemma 2.1 $\mathcal{S}_0 \subset \mathcal{S}$.

Lemma 2.2 *Assume that* v_{λ_n} *converges to* $v \in \mathcal{S}_0$ *and that some sequence of optimal actions* $\mathbf{x}_{\lambda_n} \in \mathbf{X}_{\lambda_n}(v_{\lambda_n})$ *converges to* $\mathbf{x}$. *Then* $\mathbf{x} \in \mathbf{X}_0(v)$.

Lemma 2.3 *Let* v *and* v' *be in* $\mathcal{S}$ *and* $\Omega_1 = \operatorname{Argmax}(v - v')$. *For any* $\mathbf{x} \in \mathbf{X}_0(v)$, $\mathbf{y} \in \mathbf{Y}_0(v')$, *and* $\omega \in \Omega_1$, *the probability* $\rho(\mathbf{x}, \mathbf{y}, \omega)$ *is supported by* Ω_1.

Proof Since $v \in \mathcal{S}$ and $x \in \mathbf{X}_0(v)$:

$$v(\omega) = \Phi(0, v)(\omega) \leq \mathsf{E}_{\rho(\mathbf{x},\mathbf{y},\omega)} v(\cdot).$$

Using a dual inequality as well:

$$v(\omega) - v'(\omega) \leq \mathsf{E}_{\rho(\mathbf{x},\mathbf{y},\omega)}\big(v - v'\big)(\cdot),$$

and the result follows. □

3 Absorbing Games

We consider here a special class of stochastic games, as defined in Sect. 2. We are given two separately continuous (payoff) functions g, g^* from $I \times J$ to $[-1, 1]$, and a separately continuous (probability of absorption) function p from $I \times J$ to $[0, 1]$.

The repeated game with absorbing states is played in discrete time as follows. At stage $t = 1, 2, \ldots$ (if absorption has not yet occurred) player 1 chooses $i_t \in I$ and, simultaneously, player 2 chooses $j_t \in J$:

(i) the payoff at stage t is $g(i_t, j_t)$;
(ii) with probability $p^*(i_t, j_t) := 1 - p(i_t, j_t)$, absorption is reached and the payoff in all future stages $s > t$ is $g^*(i_t, j_t)$;
(iii) with probability $p(i_t, j_t)$, the situation is repeated at stage $t + 1$.

Recall that the asymptotic analysis for these games is due to Kohlberg [2] in the case where I and J are finite.

As usual, denote $X := \Delta(I)$ and $Y := \Delta(J)$; g, p and p^* are bilinearly extended to $X \times Y$. Let $p^*(x, y)\overline{g}^*(x, y) := \int_{I\times J} p^*(i, j)g^*(i, j)x(\mathrm{d}i)y(\mathrm{d}j)$. $\overline{g}^*(x, y)$ is thus the expected absorbing payoff, conditionally to absorption.

The Shapley operator of the game is then defined on $\mathbb{R}$ by

$$\begin{aligned}\Phi(\lambda, f) &:= \min_{y\in Y}\max_{x\in X}\big\{\lambda g(x, y) + (1-\lambda)\big(p(x, y)f + p^*(x, y)\overline{g}^*(x, y)\big)\big\} \\ &:= \max_{x\in X}\min_{y\in Y}\big\{\lambda g(x, y) + (1-\lambda)\big(p(x, y)f + p^*(x, y)\overline{g}^*(x, y)\big)\big\}.\end{aligned}$$

In this framework, we can prove a stronger version of Lemma 2.3:

Lemma 3.1

(i) *Let* $f \in \mathbb{R}$ *such that* $f \geq \Phi(0, f)$ *and* $y \in Y_0(f)$. *Then, for any* $x \in X$,

$$p^*(x, y) > 0 \implies f \geq \overline{g}^*(x, y).$$

(ii) *Let* $f \in \mathbb{R}$ *such that* $f \le \Phi(0, f)$ *and* $x \in X_0(f)$. *Then, for any* $y \in Y$,

$$p^*(x, y) > 0 \implies f \le \overline{g}^*(x, y).$$

Proof We prove (i). Given $x \in X$ and $y \in Y_0(f)$,

$$f \ge \Phi(0, f) \ge p(x, y)f + p^*(x, y)\overline{g}^*(x, y)$$

and $p(x, y) = 1 - p^*(x, y)$, hence the result. □

Given $\lambda \in]0, 1[$, $x \in X$ and $y \in Y$, let $r_\lambda(x, y)$ be the induced payoff in the discounted game by the corresponding stationary strategies: $r_\lambda(x, y) := \mathsf{E}_{x,y} \sum \lambda(1 - \lambda)^{t-1} g_t$.

Lemma 3.2

$$r_\lambda(x, y) \le \begin{cases} g(x, y), & \textit{if } p^*(x, y) = 0, \\ \max(g(x, y), \overline{g}^*(x, y)), & \textit{if } p^*(x, y) > 0. \end{cases}$$

Proof

$$r_\lambda(x, y) = \lambda g(x, y) + (1 - \lambda)\big[p(x, y)r_\lambda(x, y) + p^*(x, y)\overline{g}^*(x, y)\big];$$

hence

$$r_\lambda(x, y) = \frac{\lambda g(x, y) + (1 - \lambda)p^*(x, y)\overline{g}^*(x, y)}{\lambda + (1 - \lambda)p^*(x, y)}.$$ □

The previous lemma implies the following.

Lemma 3.3 *Let* $\lambda \in]0, 1[$, $x_\lambda \in X_\lambda(v_\lambda)$ *and* $y \in Y$; *then*

$$v_\lambda \le \begin{cases} g(x_\lambda, y), & \textit{if } p^*(x_\lambda, y) = 0, \\ \max(g(x_\lambda, y), \overline{g}^*(x_\lambda, y)), & \textit{if } p^*(x_\lambda, y) > 0. \end{cases}$$

Proof Since x_λ is optimal in the discounted game, for any $y \in Y$,

$$v_\lambda \le r_\lambda(x_\lambda, y)$$

and the assertion follows from Lemma 3.2. □

Combining the preceding lemmas yields the following.

Proposition 3.1 *Assume that* $v_{\lambda_n} \to v$ *and* $x_{\lambda_n} \to x$ *with* $x_{\lambda_n} \in X_{\lambda_n}(v_{\lambda_n})$. *Let* v' *such that* $v' \ge \Phi(0, v')$ *and* $y \in Y_0(v')$; *then*

$$v \le \max\big(g(x, y), v'\big).$$

Proof For any n and any $y \in Y$, Lemma 3.3 implies that either $v_{\lambda_n} \leq g(x_{\lambda_n}, y)$ or that $p^*(x_{\lambda_n}, y) > 0$, and $v_{\lambda_n} \leq \max(g(x_{\lambda_n}, y), \overline{g}^*(x_{\lambda_n}, y))$. In the second case, since $y \in Y_0(v')$, the first assertion in Lemma 3.1 ensures that $\overline{g}^*(x_{\lambda_n}, y) \leq v'$, so in both cases we get the inequality $v_{\lambda_n} \leq \max(g(x_{\lambda_n}, y), v')$. Passing to the limit yields the result. □

Corollary 3.1 *v_λ converges as λ goes to* 0.

Proof Suppose, on the contrary, that there are two sequences $v_{\lambda_n} \to v$ and $v_{\lambda'_n} \to v'$ with $v > v'$. Up to an extraction, one can assume that $x_{\lambda_n} \in X_{\lambda_n}(v_{\lambda_n})$ converges to x and, similarly, $y_{\lambda'_n} \in Y_{\lambda'_n}(v_{\lambda'_n})$ converges to y. By Lemma 2.2, $v' = \Phi(0, v')$ and $y \in Y_0(v')$, so applying Proposition 3.1 we get $v \leq \max(g(x, y), v')$, hence $v \leq g(x, y)$. A dual reasoning yields $v' \geq g(x, y)$, a contradiction. □

We now identify the limit v of the absorbing game.

Definition 3.1 Define the function $W : X \times Y \to \mathbb{R}$ by

$$W(x, y) := \operatorname{med}\Big(g(x, y), \sup_{x';\, p^*(x', y) > 0} \overline{g}^*\big(x', y\big), \inf_{y';\, p^*(x, y') > 0} \overline{g}^*\big(x, y'\big)\Big),$$

where $\operatorname{med}(\cdot, \cdot, \cdot)$ denotes the median of three numbers, with the usual convention that a supremum (resp., an infimum) over an empty set equals $-\infty$ (resp., $+\infty$).

Corollary 3.2 *The limit v is the value of the zero-sum game, denoted by Υ, with action spaces X and Y and payoff W.*

Proof It is enough to show that $v \leq w := \sup_x \inf_y W(x, y)$ as a dual argument yields the conclusion. Assume, by contradiction, that $w < v$.

Let $\varepsilon > 0$ with $w + 2\varepsilon < v$. Consider $x \in X_0(v)$ an accumulation point of $x_\lambda \in X_\lambda(v_\lambda)$ and let y be an ε-best response to x in the game Υ. Lemma 3.1(ii) implies that

$$\inf_{y';\, p^*(x, y') > 0} \overline{g}^*\big(x, y'\big) \geq v > w + \varepsilon \geq W(x, y),$$

so that

$$W(x, y) = \max\Big(g(x, y), \sup_{x';\, p^*(x', y) > 0} \overline{g}^*\big(x', y\big)\Big).$$

Thus, $\sup_{x';\, p^*(x', y) > 0} \overline{g}^*(x', y) \leq w + \varepsilon < v - \varepsilon$ and, similarly, $g(x, y) < v - \varepsilon$. The corresponding inequalities hold with x_λ, for λ small enough:

$$p^*(x_\lambda, y)\big[\overline{g}^*(x_\lambda, y) - (v - \varepsilon)\big] \leq 0, \qquad g(x_\lambda, y) \leq v - \varepsilon,$$

leading by Lemma 3.2 to $v_\lambda \leq v - \varepsilon$, a contradiction. □

Remark 3.1 The proof of Corollary 3.2 establishes in itself the existence of the limit v (by doing the same reasoning with any accumulation point of v_λ).

Furthermore, notice that this proves that the game Υ has a value, which is not obvious a priori.

4 Recursive Games

Recursive games are another special class of stochastic games, as defined in Sect. 2. We are given a finite set $\Omega = \Omega_0 \cup \Omega^*$, two compact metric sets I and J, a payoff function g^* from Ω^* to $\mathbb{R}$, and a separately continuous function ρ from $I \times J \times \Omega_0$ to $\Delta(\Omega)$. Ω^* is the set of absorbing states, while Ω_0 is the set of recursive states.

The repeated recursive game is played in discrete time as follows. At stage $t = 1, 2, \ldots$, if absorption has not yet occurred and the current state is $\omega_t \in \Omega_0$, player 1 chooses $i_t \in I$ and, simultaneously, player 2 chooses $j_t \in J$:

(i) the payoff at stage t is 0;
(ii) the state ω_{t+1} is chosen with probability distribution $\rho(\omega_{t+1}|i_t, j_t, \omega_t)$;
(iii) if $\omega_{t+1} \in \Omega^*$, absorption is reached and the payoff in all future stages $s > t$ is $g^*(\omega_{t+1})$;
(iv) if $\omega_{t+1} \in \Omega_0$, absorption is not reached and the game continues.

The study of those recursive games was first done by Everett [5], who proved that the game has a value when considering the asymptotic payoff on plays.

As before, denote $X := \Delta(I)$ and $Y := \Delta(J)$, $\mathbf{X} := X^{\Omega}$ and, similarly, $\mathbf{Y} = Y^{\Omega}$; ρ is bilinearly extended to $\mathbf{X} \times \mathbf{Y}$. Recall that in this framework, the Shapley operator is defined from $\mathcal{F}_1 := \mathbb{R}^{\Omega_0}$ to itself by

$$\Phi(\lambda, f)(\omega) := \min_{y \in Y} \max_{x \in X} \left\{ (1-\lambda) \sum_{\omega' \in \Omega} \rho(\omega'|x, y, \omega) f\left(\omega'\right) \right\}$$

$$:= \max_{x \in X} \min_{y \in Y} \left\{ (1-\lambda) \sum_{\omega' \in \Omega} \rho(\omega'|x, y, \omega) f\left(\omega'\right) \right\},$$

where, by convention, $f(\omega') = g^*(\omega')$ whenever $\omega' \in \Omega^*$.

Proposition 4.1 *Let* $v \in \mathcal{S}_0$, *and* v' *such that* $\max_{\Omega} v(\omega) - v'(\omega) > 0$. *Assume that the inequality* $v'(\omega) \geq \Phi(0, v')(\omega)$ *holds for all* $\omega \in \Omega_1 := \operatorname{Argmax}_{\Omega}(v - v')$. *Then* $v(\cdot) \leq 0$ *on* Ω_1.

Proof Denote by Ω_2 the Argmax of v on the set Ω_1; it is enough to prove that $v(\cdot) \leq 0$ on Ω_2, so we assume the contrary. Up to extraction, $v_{\lambda_n} \to v$, $\mathbf{x}_{\lambda_n} \in \mathbf{X}_{\lambda_n}(v_{\lambda_n}) \to \mathbf{x}$ and there exists $\omega_0 \in \Omega_2$, which realizes the maximum of v_{λ_n} on Ω_2 for every n. In particular, $v(\omega_0) > 0$. Since $\mathbf{x}_{\lambda_n}$ is optimal, we get, for any $\mathbf{y} \in \mathbf{Y}$:

$$v_{\lambda_n}(\omega_0) \leq (1-\lambda_n) \left[\sum_{\omega' \in \Omega_2} \rho(\omega'|\mathbf{x}_{\lambda_n}, \mathbf{y}, \omega_0) v_{\lambda_n}\left(\omega'\right) + \sum_{\omega' \in \Omega \setminus \Omega_2} \rho(\omega'|\mathbf{x}_{\lambda_n}, \mathbf{y}, \omega_0) v_{\lambda_n}\left(\omega'\right) \right],$$

so, by definition of ω_0,

$$\big(1-(1-\lambda_n)\rho(\Omega_2|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)\big)v_{\lambda_n}(\omega_0) \le (1-\lambda_n)\sum_{\omega'\in\Omega\setminus\Omega_2}\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)v_{\lambda_n}\big(\omega'\big).$$

For simplicity, denote $\rho_n := \rho(\Omega_2|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)$. If $\rho_n = 1$ for infinitely many n, we immediately get $v(\omega_0)\le 0$ and the requested contradiction, hence we assume that it is not the case. Hence, up to an extraction, μ_n defined by $\mu_n(w') = \frac{\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)}{1-\rho_n}$ is a probability measure on $\Omega\setminus\Omega_2$. Then, for n large enough, we get an analogue of Lemma 3.3:

$$v_{\lambda_n}(\omega_0) \le \frac{1-\lambda_n}{1-(1-\lambda_n)\rho_n}\sum_{\omega'\in\Omega\setminus\Omega_2}\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)v_{\lambda_n}\big(\omega'\big) \tag{1}$$

$$= \frac{(1-\lambda_n)(1-\rho_n)}{\lambda_n+(1-\lambda_n)(1-\rho_n)}\sum_{\omega'\in\Omega\setminus\Omega_2}\frac{\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)}{1-\rho_n}v_{\lambda_n}\big(\omega'\big) \tag{2}$$

$$\le \max\Big(0, \sum_{\omega'\in\Omega\setminus\Omega_2}\mu_n\big(\omega'\big)v_{\lambda_n}\big(\omega'\big)\Big). \tag{3}$$

On the other hand, choose now $\mathbf{y}\in\mathbf{Y}_0(v')$. Since $\omega_0\in\Omega_2$,

$$\begin{aligned} v'(\omega_0) &\ge \Phi\big(0,v'\big)(\omega_0) \\ &\ge \Big[\sum_{\omega'\in\Omega_2}\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)v'\big(\omega'\big) + \sum_{\omega'\in\Omega\setminus\Omega_2}\rho(\omega'|\mathbf{x}_{\lambda_n},\mathbf{y},\omega_0)v'\big(\omega'\big)\Big], \end{aligned}$$

so using the fact that v' is constant on Ω_2, we get an analogue to Lemma 3.1:

$$v'(\omega_0) \ge \sum_{\omega'\in\Omega\setminus\Omega_2}\mu_n\big(\omega'\big)v'\big(\omega'\big). \tag{4}$$

Letting n go to infinity in inequalities (3) and (4), and using $v(\omega_0)>0$, we obtain by compactness the existence of $\mu\in\Delta(\Omega\setminus\Omega_2)$ such that

$$v(\omega_0) \le \sum_{\omega'\in\Omega\setminus\Omega_2}\mu\big(\omega'\big)v\big(\omega'\big), \tag{5}$$

$$v'(\omega_0) \ge \sum_{\omega'\in\Omega\setminus\Omega_2}\mu\big(\omega'\big)v'\big(\omega'\big). \tag{6}$$

Subtracting (6) from (5) yields

$$\big(v-v'\big)(\omega_0) \le \sum_{\omega'\in\Omega\setminus\Omega_2}\mu\big(\omega'\big)\big(v-v'\big)\big(\omega'\big),$$

and since $\omega_0\in\Omega_1 = \mathrm{Argmax}_{\Omega}(v-v')$, this implies that the support of μ is included in Ω_1 and that (5) is an equality. This, in turn, forces the support of μ to be included in $\Omega_2 = \mathrm{Argmax}_{\Omega_1} v$, a contradiction to the construction of μ. □

Corollary 4.1 *v_λ converges as λ goes to* 0.

Proof Assume that there are two accumulation points v and v' with $\max_\Omega\{v - v'\} > 0$, and denote $\Omega_1 = \operatorname{Argmax}_\Omega(v - v')$. Then Proposition 4.1 implies that $v(\cdot) \le 0$ on Ω_1. A dual argument yields $v'(\cdot) \ge 0$ on Ω_1, a contradiction. □

We now recover a characterization of the limit due to Everett [5]:

Corollary 4.2 *$\mathcal{S}_0 \subset \overline{\mathscr{L}^+} \cap \overline{\mathscr{L}^-}$, where $\overline{A}$ is the closure of A and*

$$\mathscr{L}^+ := \left\{ f \in \mathbb{R}^\Omega, \quad \begin{array}{ll} \Phi(0, f)(\omega) \le f(\omega) & \forall \omega \in \Omega_0 \\ \Phi(0, f)(\omega) = f(\omega) & \implies f(\omega) \ge 0 \\ f(\omega) \ge g^*(\omega) & \forall \omega \in \Omega^* \end{array} \right\}, \tag{7}$$

and symmetrically

$$\mathscr{L}^- := \left\{ f \in \mathbb{R}^\Omega, \quad \begin{array}{ll} \Phi(0, f)(\omega) \ge f(\omega) & \forall \omega \in \Omega_0 \\ \Phi(0, f)(\omega) = f(\omega) & \implies f(\omega) \le 0 \\ f(\omega) \le g^*(\omega) & \forall \omega \in \Omega^* \end{array} \right\}. \tag{8}$$

We will need the following lemma.

Lemma 4.1 *For any $\varepsilon \ge 0$, there exist $\Omega' \subset \Omega_0$ and $v' \in \mathcal{F}_1$ such that the couple (Ω', v') satisfies*

(a) *$v'(\omega) = g^*(\omega)$ for all $\omega \in \Omega^*$.*
(b) *$v'(\omega) = v(\omega) - \varepsilon$ on Ω'.*
(c) *$v(\omega) \ge v'(\omega) > v(\omega) - \varepsilon$ on $\Omega_0 \backslash \Omega'$.*
(d) *For any $\omega \in \Omega_0 \backslash \Omega'$, $\Phi(0, v')(\omega) > v'(\omega)$.*
(e) *For any $\omega \in \Omega'$, $\Phi(0, v')(\omega) = v'(\omega)$.*

Proof This was proved in [10], but we recall the proof for the sake of completeness.

Let $\mathscr{E}$ be the set of couples (Ω'', v'') such that $\Omega'' \subset \Omega_0$, $v'' \in \mathcal{F}_1$, and (Ω'', v'') satisfies properties (a) to (d). This set is nonempty since $(\Omega_0, v - \varepsilon \mathbb{1}_{\omega \in \Omega_0}) \in \mathscr{E}$. Since Ω_0 is finite, we can choose a couple (Ω', v') in $\mathscr{E}$ such that there is no (Ω'', v'') in $\mathscr{E}$ with $\Omega'' \subsetneq \Omega'$. Let $\widetilde{\Omega}$ be the set on which $\Phi(0, v')(\omega) = v'(\omega)$; we now prove that $\widetilde{\Omega} = \Omega'$, hence that (Ω', v') also satisfies property (e).

By contradiction, assume that $\widetilde{\Omega} \subsetneq \Omega'$ and consider, for small $\alpha > 0$, $v_\alpha := v' + \alpha \mathbb{1}_{\omega \in \Omega' \backslash \widetilde{\Omega}}$. The couple $(\widetilde{\Omega}, v_\alpha)$ clearly satisfies properties (a) to (c) for $\alpha < \varepsilon$. It also satisfies property (d) for α small enough by continuity of $\Phi(0, \cdot)$. So, for α small enough, the couple $(\widetilde{\Omega}, v_\alpha)$ is in $\mathscr{E}$, contradicting the minimality of Ω'. □

We can now prove Corollary 4.2:

Proof of Corollary 4.2 Let $v \in \mathcal{S}_0$, let $\varepsilon > 0$ and define (v', Ω') as in Lemma 4.1. By properties (a) to (c), $\|v - v'\|_\infty \le \varepsilon$. If $\Omega' = \emptyset$, then property (d) implies that $v' \in \mathscr{L}^-$. If Ω' is nonempty, then, by properties (b), (c) and (e), $\Omega' = \operatorname{Argmax}(v - v')$ and

$\Phi(0, v')(\cdot) = v'(\cdot)$ on Ω'. Hence, Proposition 4.1 yields $v(\cdot) \le 0$ on Ω'. So $v'(\cdot) \le 0$ on Ω' and $v' \in \mathcal{L}^-$ as well. This implies that $v \in \overline{\mathcal{L}^-}$. By duality, $v \in \overline{\mathcal{L}^+}$. □

Remark 4.1 This corollary implies in itself that v_λ converges, as there is at most one element in the intersection, see [3] and Proposition 9 in [6].

5 Games with Incomplete Information

We consider here two person zero-sum games with incomplete information (independent case and standard signaling). π is a product probability $p \otimes q$ on a finite product space $K \times L$, with $p \in P = \Delta(K)$, $q \in Q = \Delta(L)$. g is a payoff function from $I \times J \times K \times L$ to $\mathbb{R}$ where I and J are finite action sets. Given the parameter (k, ℓ) selected according to π, each player knows one component (k for player 1, ℓ for player 2) and holds a prior on the other component. From stage 1 on, the parameter is fixed, the repeated game with payoff $g(\cdot, \cdot, k, \ell)$ is played. The moves of the players at stage t are $\{i_t, j_t\}$, the payoff is $g_t = g(i_t, j_t, k, \ell)$ and the information of the players after stage t is $\{i_t, j_t\}$. $\overline{X} = \Delta(I)^K$ and $\overline{Y} = \Delta(J)^L$ are the type-dependent mixed action sets of the players; g is extended on $\overline{X} \times \overline{Y} \times K \times L$ by $g(x, y, p, q) = \sum_{k,\ell} p^k q^\ell g(x^k, y^\ell, k, \ell)$.

Given (x, y, p, q), let $x(i) = \sum_k x_i^k p^k$ be the probability of action i and $p(i)$ be the conditional probability on K given the action i, explicitly $p^k(i) = \frac{p^k x_i^k}{x(i)}$ (and, similarly, for y and q).

While this framework is not a particular case of Sect. 2, since the set $P \times Q$ that will play the role of the state space is not finite, it is still possible to introduce a Shapley operator for this game. This operator is defined on the set $\mathcal{F}_2$ of continuous concave-convex functions on $P \times Q$ by

$$\Phi(\lambda, f)(p, q) := \min_{y \in \overline{Y}} \max_{x \in \overline{X}} \left\{ \lambda g(p, q, x, y) + (1 - \lambda) \sum_{i,j} x(i) y(j) f\big(p(i), q(j)\big) \right\} \tag{9}$$

$$:= \max_{x \in \overline{X}} \min_{y \in \overline{Y}} \left\{ \lambda g(p, q, x, y) + (1 - \lambda) \sum_{i,j} x(i) y(j) f\big(p(i), q(j)\big) \right\} \tag{10}$$

and the value v_λ of the λ-discounted game is the unique fixed point of $\Phi(\lambda, .)$ on $\mathcal{F}_2$. These relations are due to Aumann and Maschler (1966) [7] and Mertens and Zamir (1971) [8].

$\overline{X}_\lambda(f)(p, q)$ denotes the set of optimal strategies of player 1 in $\Phi(\lambda, f)(p, q)$.

In this framework, any $f \in \mathcal{F}_2$ is a fixed point of the projective operator $\Phi(0, .)$, that is, $\mathcal{F}_2 = \mathcal{S}$.

Note that, if C is a bound for the payoff function g, then any v_λ is bounded by C as well, and is moreover C-Lipschitz. The family $\{v_\lambda\}$ is thus relatively compact for the topology of uniform convergence, hence $\mathcal{S}_0$, the set of accumulation points of the family $\{v_\lambda\}$, is nonempty.

To ease the notations, we will denote the sum $\sum_{i,j} x(i)y(j)f(p(i),q(j))$ by $\mathsf{E}_{\rho(x,p)\times\rho'(y,q)} f(\tilde{p},\tilde{q})$. Note that $\tilde{p}$ only depends on x and p, and that $\tilde{q}$ only depends on y and q.

For any $f \in \mathcal{F}_2$, Jensen's inequality ensures that $\mathsf{E}_{\rho(x,p)} f(\tilde{p},q) \leq f(p,q)$. The strategies of player 1 for which the equality holds for all $f \in \mathcal{F}_2$ are called non-revealing. Their set is denoted $NR(p) := \{x \in X;\ \tilde{p} = p, \rho(x,p)\ a.s.\}$. The set $NR(q)$ of non-revealing strategies of player 2 is defined similarly.

Finally, the non-revealing value u is

$$u(p,q) := \min_{y\in NR(q)} \max_{x\in NR(p)} g(x,y,p,q) = \max_{x\in NR(p)} \min_{y\in NR(q)} g(x,y,p,q).$$

The existence of $\lim v_\lambda$ was first proved in [7] for games with incomplete information on one side. It was then generalized in [8] for games with incomplete information on both sides, with a characterization of the limit v being the only solution of the system

$$v = \mathtt{Cav}_p \min(u,v),\ \ v = \mathtt{Vex}_q \max(u,v),$$

where $\mathtt{Cav}(f)$ (resp. $\mathtt{Vex}(f)$) denotes the smallest concave function in the first variable which is larger than f (resp., the largest convex function in the second variable which is smaller than f).

A shorter proof of this result (including characterization) was established in [9]. The tools used in the following proof are quite similar to the one used in [9], but the structure differs.

Lemmas 2.1 and 2.2 still hold in this framework; we now prove a more precise version of Lemma 2.3 using the geometry of $P\times Q$. Let $\mathscr{C}(P\times Q)$ be the set of real continuous functions on $P\times Q$.

Lemma 5.1 *Let $v \in \mathcal{S}$ and let $f \in \mathscr{C}(P\times Q)$ be concave with respect to the first variable. If (p,q) is an extreme point of* Argmax$(v-f)$, *then $\overline{X}_0(v)(p,q) \subset NR(p)$.*

Proof Let $x \in \overline{X}_0(v)(p,q)$ and $y \in NR(q)$; then

$$v(p,q) \leq \mathsf{E}_{\rho(x,p)\times\rho'(y,q)} v(\tilde{p},\tilde{q}) = \mathsf{E}_{\rho(x,p)} v(\tilde{p},q),$$

while, by Jensen's inequality,

$$f(p,q) \geq \mathsf{E}_{\rho(x,p)} f(\tilde{p},q);$$

so

$$\mathsf{E}_{\rho(x,p)}(v-f)(\tilde{p},q) \geq (v-f)(p,q).$$

Since $(p,q) \in \text{Argmax}(v-f)$,

$$\mathsf{E}_{\rho(x,p)}(v-f)(\tilde{p},q) = (v-f)(p,q),$$

and $(\tilde{p}, q) \in \text{Argmax}(v - f)$, $\rho(x, p)$ *a.s.* Since (p, q) is an extreme point of $\text{Argmax}(v - f)$, it follows that $\tilde{p} = p$, $\rho(x, p)$ *a.s.* and $x \in NR(p)$. □

Remark that $v \in \mathcal{F}_2$ implies that $NR(p) \subset \overline{X}_0(v)(p, q)$ since v is a saddle function; hence, in fact, $NR(p) = \overline{X}_0(v)(p, q)$ in the previous lemma.

Note the analogy between Lemma 5.1 and Lemma 3.1. Lemma 3.3 also has an analogue in this setup:

Lemma 5.2 *Let* $x_\lambda \in \overline{X}_\lambda(v_\lambda)(p, q)$ *and* $y \in NR(q)$, *then*

$$v_\lambda(p, q) \leq g(x_\lambda, y, p, q).$$

Proof By definition of v_λ and x_λ,

$$\begin{aligned} v_\lambda(p, q) &\leq \lambda g(x_\lambda, y, p, q) + (1 - \lambda)\mathsf{E}_{\rho(x_\lambda, p) \times \rho'(y, q)} v_\lambda(\tilde{p}, \tilde{q}) \\ &\leq \lambda g(x_\lambda, y, p, q) + (1 - \lambda) v_\lambda(p, q), \end{aligned}$$

using Jensen's inequality and the fact that $y \in NR(q)$. Hence $v_\lambda(p, q) \leq g(x_\lambda, y, p, q)$. □

Recall that $\mathcal{S}_0 \subset \mathcal{S}$ is the set of accumulation points of $\{v_\lambda\}$ for the uniform norm.

Proposition 5.1 *Let* $v \in \mathcal{S}_0$.

(i) *Let* $f \in \mathscr{C}(P \times Q)$ *be concave with respect to the first variable. Then, at any extreme point* (p, q) *of* $\text{Argmax}(v - f)$,

$$v(p, q) \leq u(p, q).$$

(ii) *Let* $f' \in \mathscr{C}(P \times Q)$ *be convex with respect to the second variable. Then, at any extreme point* (p, q) *of* $\text{Argmin}(v - f')$,

$$v(p, q) \geq u(p, q).$$

Proof We prove (i). Apply Lemma 5.2 to any sequence $\{v_{\lambda_n}\}$ converging to v. By Lemma 5.2, there exists $x \in \overline{X}_0(v)(p, q)$ such that

$$v(p, q) \leq \inf_{y \in NR(q)} g(x, y, p, q).$$

Lemma 5.1 implies that $x \in NR(p)$ (since $v \in \mathcal{S}$), and the result follows by definition of u.

(ii) is established in a dual way. □

Proposition 5.1 implies the following corollaries of existence and characterization of $\lim v_\lambda$:

Corollary 5.1 v_λ *converges uniformly as* λ *tend to* 0.

Proof Let v and v' in $\mathcal{S}_0$ and let (p,q) be any extreme point of $\text{Argmax}(v - v')$. Since v' is concave in its first variable, Proposition 5.1(i) with $f = v'$ implies that $v(p,q) \le u(p,q)$. Apply now Proposition 5.1(ii) to $f' = v$ to get $v'(p,q) \ge u(p,q)$. This yields $v(p,q) \le v'(p,q)$, hence $v \le v'$, thus uniqueness. □

Corollary 5.2 *Any accumulation point v of v_λ satisfies the Mertens–Zamir system*:

$$v = \mathtt{Cav}_p \min(u, v), \qquad v = \mathtt{Vex}_q \max(u, v).$$

Proof Let v be an accumulation point of the family $\{v_\lambda\}$. We only prove that $v \le \mathtt{Cav}_p \min(u, v)$. Since v is concave in p, the other inequality is trivial, and a dual argument gives the dual equality. Denote $f = \mathtt{Cav}_p \min(u, v)$, and let (p,q) be any extreme point of $\text{Argmax}(v - f)$. Since f is concave in p, Proposition 5.1 implies that $v(p,q) \le u(p,q)$. Hence,

$$v(p,q) \le \min(u, v)(p,q) \le f(p,q)$$

and thus $v \le f$. □

Remark 5.1

(i) The proof above also shows that v is the smallest among the functions satisfying $w = \mathtt{Cav}_p \min(u, w)$.
(ii) A similar approach applies word for word to the dependent case, as defined in Mertens and Zamir [8].
(iii) The case where the action sets I and J are compact metric can also be handled in the same way, using the martingales $(\tilde{p}, \tilde{q})$ of regular conditional probabilities.

6 Conclusion

This paper proposes a unified proof of existence of the limit of the discounted value for three families of zero-sum repeated games. The proofs are based on the Shapley operator and the associated fixed point.

Recall that a similar formula holds for the discounted value of general zero-sum repeated games ([11], Chap. 4). Hence we expect to extend the current approach to further classes of games.

It could also be used for other evaluations of the payoff beyond the discounted case, for example, to prove the convergence of the value of the n-stage game when n tends to infinity.

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